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AN ABSTRACT NONLINEAR VOLTERRA EQUATION

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ABSTRACT

The existence, uniqueness, regularity and dependence upon data of solutions of the abstract Volterra equation

$$u(t)+\int_0^t a(t-s)A(u(s))ds \ni f(t), \qquad t \geqq 0$$

are studied in a real Banach space. The nonlinear operator A is assumed to be m-accretive and the assumptions on the kernel a do not exclude the possibility that $\lim_{t\to 0+} a(t) = +\infty$.

1. Introduction and statement of results

The purpose of this paper is to study the existence, uniqueness, regularity and dependence upon data of solutions of the Volterra equation

$$(1.1) \qquad u(t)+\int_0^t a(t-s)A(u(s))ds \ni f(t), \qquad t \in R^+ = [0,\infty)$$

in a real Banach space X. It is assumed that A is essentially an m-accretive operator in X (see [2] for definitions), but the main point of interest is that the assumptions on a that we use, do not exclude the possibility that $\lim_{t\to 0+} a(t) = +\infty$. Hence it is not in general possible under our assumptions to reduce (1.1) to an initial value problem of the form

$$\begin{cases} du/dt + A(u) \ni G(u), & t \in R^+, \\ u(0) = x \end{cases}$$

where G satisfies certain Lipschitz conditions, cf. [6]. Therefore the results of this paper generalize those of [6] and [7]. Under assumptions on the kernel that

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are related to the ones used here (but which also allow the kernel to be operator-valued in a specific way) the problem of existence of solutions of (1.1) has been studied in [3], in a Hilbert space setting when A is the subdifferential of a convex function. For other existence results on (1.1) when X is a Hilbert space, that are not covered by this paper, see [1], [9], [10], [14]. Equation (1.1) has also been studied in [11], [12], [13] under different assumptions.

Our first result is

THEOREM 1. *Assume that*

$$X \text{ is a real reflexive Banach space with locally uniformly convex dual } X^*, \tag{1.2}$$

$$A = B + \omega I \text{ where } B \subset X \times X \text{ is } m\text{-accretive, } \omega \text{ is a real number and } I \text{ is the identity mapping,} \tag{1.3}$$

$$a(t) = b(t) + c(t), \quad t > 0 \text{ where} \tag{1.4}$$

$$b \in L^1_{\text{loc}}(R^+; R) \cap C^1((0,\infty); R), \tag{1.5}$$

$$b(t) > 0 \text{ and } b \text{ is nonincreasing when } t > 0, \tag{1.6}$$

$$\log(b(t)) \text{ is convex on } (0,\infty), \tag{1.7}$$

$$c \in AC_{\text{loc}}(R^+; R), \qquad c(0) = 0, \tag{1.8}$$

$$c' \in BV_{\text{loc}}(R^+; R), \tag{1.9}$$

$$f \in W^{1,1}_{\text{loc}}(R^+; X), \tag{1.10}$$

$$f' \in BV_{\text{loc}}(R^+; X), \tag{1.11}$$

$$f(0) \in D(B). \tag{1.12}$$

Then there exists a unique function

$$u \in C(R^+; \overline{D(B)}) \cap BV_{\text{loc}}(R^+; \overline{D(B)}) \tag{1.13}$$

such that

$$u(t) + \int_0^t a(t-s)w(s)ds = f(t), \qquad t \in R^+, \tag{1.14}$$

where

$$w \in L^\infty_{\text{loc}}(R^+; X) \tag{1.15}$$

is such that

$$[u(t), w(t) - \omega u(t)] \in B, \qquad \text{a.e. } t \in R^+. \tag{1.16}$$

If $\lim_{t\to 0+} a(t) < +\infty$ *then it is not necessary to assume that* X^* *is locally uniformly convex.*

Here $f \in W^{1,1}_{\text{loc}}(R^+; X)$ means that $f(t) = f(0) + \int_0^t f'(s)ds$, $f' \in L^1_{\text{loc}}(R^+; X)$, and $D(B) = \{x \mid [x, y] \in B \text{ for some } y \in X\}$.

In some cases when we cannot show that there exists a solution of (1.1) in the sense (1.13)–(1.16) we can at least show that the solutions of certain approximating equations converge to a function that may be considered to be a "weak" solution of (1.1).

THEOREM 2. *Assume that* (1.3)–(1.10) *hold and that*

$$X \text{ is a real Banach space,} \tag{1.17}$$

$$f(0) \in \overline{D(B)}. \tag{1.18}$$

If u_λ, $\lambda > 0$, *is the solution of the equation*

$$u_\lambda(t) + \int_0^t a(t-s)(B_\lambda(u_\lambda(s)) + \omega u_\lambda(s))ds = f(t), \qquad t \in R^+ \tag{1.19}$$

then there exists a function

$$u \in C(R^+; \overline{D(B)}) \tag{1.20}$$

such that

$$u_\lambda \to u \quad \text{as } \lambda \to 0 \text{ uniformly on compact subsets of } R^+. \tag{1.21}$$

Moreover, if (1.11) *holds and*

$$\sup_{\lambda > 0} \| B_\lambda(f(0)) \| < \infty \tag{1.22}$$

then (1.13) *holds. If the assumptions of Theorem* 1 *hold, then the function u is the solution satisfying* (1.13)–(1.16).

The Yosida approximation B_λ is defined by $B_\lambda = \lambda^{-1}(I - (I + \lambda B)^{-1})$ and is Lipschitz-continuous if B is m-accretive. Note that (1.12) implies (1.22), but the converse need not hold unless X is reflexive ($\|\cdot\|$ denotes the norm in X).

The last theorem shows how the solutions depend upon the function f.

THEOREM 3. *Assume that* (1.3)–(1.9) *and* (1.17) *hold. Then there exists a nondecreasing, continuous function* k *on* R^+, $k(0)=1$, *such that if*

$$f_i \in W^{1,1}_{\mathrm{loc}}(R^+;X), \qquad i=1,2, \tag{1.23}$$

$$f_i(0) \in \overline{D(B)}, \qquad i=1,2 \tag{1.24}$$

and u_i, $i=1,2$ *are the limit functions corresponding to* f_i (*that exist by Theorem* 2), *then*

$$\|u_1(t)-u_2(t)\| \leqq k(t)\Big(\|f_1(0)-f_2(0)\| + \int_0^t \|f_1'(s)-f_2'(s)\|ds\Big), \quad t \in R^+ \tag{1.25}$$

Moreover, if $c \equiv 0$ *and* $\omega = 0$, *then* $k \equiv 1$.

Observe that the assumptions (1.5)–(1.7) are satisfied if b is locally integrable and completely monotone. Thus one may, for example, take $a(t)=t^{-\alpha}$, $t>0$, $0<\alpha<1$, in (1.1), and in this case (1.1) corresponds to a differential equation of fractional order $1-\alpha$.

For results on the asymptotic behaviour of the solutions of (1.1), see [4] (but note that the assumption $\lim_{t\to 0+} a(t) < +\infty$ that is made in [4] is not necessary for the proof as long as one knows that the functions u_λ converge).

2. Proof of Theorem 1

First we establish some results concerning linear Volterra integral equations that will be used later. Define the function b_n (n is a positive integer) by $b_n(t)=b(t+n^{-1})$, $t\in R^+$ and the function a_n by $a_n(t)=b_n(t)+c(t)$, $t\in R^+$. Let p_n and P_n be the solutions of the equations

$$b_n(0)p_n(t) = -b_n'(t) - \int_0^t b_n'(t-s)p_n(s)ds, \qquad t\in R^+ \tag{2.1}$$

and

$$b_n(0)P_n(t) = -a_n'(t) - \int_0^t a_n'(t-s)P_n(s)ds, \qquad t\in R^+. \tag{2.2}$$

LEMMA 2.1. *Assume that* (1.4)–(1.9), (2.1) *and* (2.2) *hold. Then*

(2.3) $p_n \in BV(R^+;R)$ *is nonnegative and nonincreasing on* R^+,

there exists a nonnegative and nonincreasing function

(2.4) $p \in L^1_{\mathrm{loc}}(R^+; R)$ *such that* $b_n(0)^{-1}p_n \to p$ *in* $L^1(0, T; R)$ *as* $n \to \infty$

$$\text{for all } T > 0 \text{ and } b(0)^{-1}b(t) + \int_0^t p(t-s)b(s)ds = 1, \quad t > 0,$$

$$P_n = p_n - b_n(0)q_n \text{ where } q_n \in BV_{\mathrm{loc}}(R^+; R)$$

(2.5) *and there exists* $q \in BV_{\mathrm{loc}}(R^+; R)$ *such that*

$$|q_n(0) - q(0)| + \mathrm{var}(q_n - q; [0, T]) \to 0 \text{ as } n \to \infty \text{ for all } T > 0.$$

PROOF. Let R_μ be the solution of the equation

$$R_\mu(t) + \mu \int_0^t b(t-s)R_\mu(s)ds = 1, \qquad t \in R^+, \quad \mu > 0, \tag{2.6}$$

It follows from [8, lines (1.8)–(1.10)] that R_μ is positive, decreasing and satisfies the inequality

$$R_\mu(t) \leqq \left(1 + \mu \int_0^t b(s)ds\right)^{-1}, \qquad t \in R^+. \tag{2.7}$$

Hence it follows from Helly's theorem that there exists a nonnegative, nonincreasing function p such that

$$\lim_{\mu_n \to \infty} \mu_n R_{\mu_n}(t) = p(t), \qquad t > 0 \text{ for some sequence } \{\mu_n\}. \tag{2.8}$$

To see that we also have $p \in L^1_{\mathrm{loc}}(R^+; R)$ we have only to note from (2.6) that

$$\sup_{\mu > 0} \int_0^T \mu R_\mu(s)ds \leqq e^{xT}\left(x \int_0^\infty e^{-xs}b(s)ds\right)^{-1}, \qquad T > 0$$

for every $x > 0$. From this inequality we also conclude that $\int_0^t \mu_n R_{\mu_n}(s)ds$ converges as $\mu_n \to \infty$ and to see that the limit function is $b(0)^{-1} + \int_0^t p(s)ds$ we recall that

$$\int_0^\infty e^{-xt} \int_0^t \mu R_\mu(s)dsdt = \left(x^2 \int_0^\infty e^{-xs}b(s)ds + x^2/\mu\right)^{-1}, \qquad x > 0.$$

If we now let $\mu_n \to \infty$ in (2.6) and use (2.7) and (2.8) we conclude that

$$b(0)^{-1}b(t) + \int_0^t p(t-s)b(s)ds = 1, \qquad t > 0. \tag{2.9}$$

If we apply this argument to the function b_n and differentiate the resulting equation (2.9) we see that (2.3) holds (the only nontrivial part of (2.3) is the statement that p_n is nonincreasing). By the argument above we know that the function p exists and has the desired properties and we have only to show that the functions $b_n(0)^{-1}p_n$ converge to p. This is easily done in the case when $\lim_{t\to 0+} b(t) < +\infty$. If $\lim_{t\to 0+} b(t) = +\infty$, then we can apply the same argument as above and use the fact (see [8, line (1.10)]) that by (1.6) and (2.1)

$$b_n(0)^{-1}\int_0^t p_n(s)ds \leqq b_n(t)^{-1} - b_n(0)^{-1}, \qquad t \in R^+.$$

To establish (2.5) we define the function d_n by

$$d_n(t) = b_n(0)^{-1}\left(c'(t) + \int_0^t c'(t-s)p_n(s)ds\right), \qquad t \in R^+ \tag{2.10}$$

and let e_n be the solution of the equation

$$e_n(t) = d_n(t) - \int_0^t d_n(t-s)e_n(s)ds, \qquad t \in R^+. \tag{2.11}$$

It is not difficult to see that

$$P_n(t) = p_n(t) - \int_0^t p_n(t-s)e_n(s)ds - e_n(t), \qquad t \in R^+. \tag{2.12}$$

From (1.9), (2.3) and (2.4) we see that $d_n \in BV_{loc}(R^+; R)$ and

$$\lim_{m,n\to\infty}(|d_m(0) - d_n(0)| + \mathrm{var}(d_m - d_n; [0, T])) = 0, \qquad T > 0$$

and hence it is straightforward to deduce from (2.11) that the same statement is true for the function e_n. But then (2.5) follows from (2.4) and (2.12), and the proof of Lemma 2.1 is completed.

Let $B_\lambda = \lambda^{-1}(I - (I + \lambda B)^{-1})$. Since B is assumed to be m-accretive it follows that B_λ is Lipschitz-continuous (with Lipschitz constant $2/\lambda$). This implies that there exists a unique solution of the equation

$$u_{\lambda,n}(t) + \int_0^t a_n(t-s)A_\lambda(u_{\lambda,n}(s))ds = f(t), \qquad t \in R^+ \tag{2.13}$$

where we use the notation $A_\lambda = B_\lambda + \omega I$. By (1.4), (1.5), (1.8)–(1.11) and (2.13) $u_{\lambda,n}$ is locally Lipschitz-continuous, differentiable a.e. and

$$u'_{\lambda,n}(t) + a_n(0)A_\lambda(u_{\lambda,n}(t)) + \int_0^t a'_n(t-s)A_\lambda(u_{\lambda,n}(s))ds = f'(t), \quad \text{a.e. } t \in R^+.$$

We solve $a_n(0)A_\lambda(u_{\lambda,n}(t))$ from this equation using the (resolvent) equation (2.2) (recall that $a_n(0)=b_n(0)$) and we obtain (see (2.5))

$$u'_{\lambda,n}(t)+\int_0^t p_n(t-s)u'_{\lambda,n}(s)ds+b_n(0)B_\lambda(u_{\lambda,n}(t))$$

$$(2.14)\qquad =-b_n(0)\omega u_{\lambda,n}(t)+b_n(0)\int_0^t q_n(t-s)u'_{\lambda,n}(s)ds+f'(t)$$

$$+\int_0^t P_n(t-s)f'(s)ds$$

$$\stackrel{\text{def}}{=} F_{\lambda,n}(t), \qquad \text{a.e. } t\in R^+.$$

From this equation we are going to derive the *a priori* estimates that we need and it will also be used in showing that the functions $u_{\lambda,n}$ converge as $n\to\infty$, $\lambda\to 0$.

If $x,y\in X$, define ($\|\cdot\|$ is the norm in X)

$$[x,y]_+=\inf_{\lambda>0}\lambda^{-1}(\|y+\lambda x\|-\|y\|), \qquad [x,y]_-=-[-x,y]_+.$$

The accretivity of B implies that B_λ is accretive and this means that

$$(2.15)\qquad [B_\lambda(x_1)-B_\lambda(x_2),x_1-x_2]_+\geqq 0, \qquad x_1,x_2\in X.$$

We also note that if $v:[0,T]\to X$ is absolutely continuous and differentiable a.e., then

$$(2.16)\qquad d/dt\,\|v(t)\|=[v'(t),v(t)]_+=[v'(t),v(t)]_-, \quad \text{a.e. } t\in[0,T]$$

(see [5, lemma 2.16]). Finally we obviously have

$$[x,x]_+=[x,x]_-=\|x\|,$$

$$[x_1+x_2,y]_+\leqq[x_1,y]_++[x_2,y]_+,$$

$$[x,y]_+\leqq\|x\|.$$

Let $h>0$ be arbitrary. By (2.14)–(2.16) we have

$$d/ds\,\|u_{\lambda,n}(s+h)-u_{\lambda,n}(s)\|+p_n(0)\|u_{\lambda,n}(s+h)-u_{\lambda,n}(s)\|$$

$$-\int_0^s |p_n(s+h-\tau)-p_n(s-\tau)|\,\|u'_{\lambda,n}(\tau)\|d\tau$$

$$\leqq\|F_{\lambda,n}(s+h)-F_{\lambda,n}(s)\|+p_n(0)\|u_{\lambda,n}(s+h)-u_{\lambda,n}(s)-hu'_{\lambda,n}(s)\|$$

$$+\|hp_n(0)u'_{\lambda,n}(s)-\int_s^{s+h}p_n(s+h-\tau)u'_{\lambda,n}(\tau)d\tau\|,\qquad \text{a.e. } s\in R^+.$$

We integrate this inequality over $(0,t)$, divide by h and let $h\to 0$. Then it follows from (2.3) and the dominated convergence theorem that

$$\|u'_{\lambda,n}(t)\|+\int_0^t p_n(t-s)\|u'_{\lambda,n}(s)\|ds$$
$$\le\|u'_{\lambda,n}(0)\|+\operatorname{var}(F_{\lambda,n};[0,t]),\qquad \text{a.e. } t\in R^+. \tag{2.17}$$

it is clear from (2.13) and (2.14) that

$$\|u'_{\lambda,n}(0)\|\le b_n(0)(\|B_\lambda(f(0))\|+|\omega|\,\|f(0)\|)+\|f'(0)\| \tag{2.18}$$

(we assume that $\lim_{t\to 0+}f'(t)=f'(0)$), and that

$$\operatorname{var}(F_{\lambda,n};[0,t])\le b_n(0)\int_0^t(|\omega|+|q_n(0)|+\operatorname{var}(q_n;[0,t-s]))\|u'_{\lambda,n}(s)\|ds$$
$$+\operatorname{var}(f';[0,t])+\int_0^t(\|f'(0)\|+\operatorname{var}(f';[0,t-s]))(p_n(s)+b_n(0)|q_n(s)|)ds, \tag{2.19}$$
$$t\in R^+.$$

Using equation (2.1) and integrating (that is, we "solve" $\|u'_{\lambda,n}(t)\|$ from the left side of (2.17) with the aid of (2.1), the integration will allow us to keep the inequality since $b_n(t)>0$), we conclude from (2.17)–(2.19) that

$$\int_0^t\|u'_{\lambda,n}(s)\|ds\le\int_0^t b_n(t-s)(\|B_\lambda(f(0))\|+|\omega|\,\|f(0)\|$$
$$+\int_0^s(|\omega|+|q_n(0)|+\operatorname{var}(q_n;[0,s-\tau]))\|u'_{\lambda,n}(\tau)\|d\tau \tag{2.20}$$
$$+\int_0^s(\|f'(0)\|+\operatorname{var}(f';[0,s-\tau]))|q_n(\tau)|\,d\tau)ds+\int_0^t(\|f'(0)\|+\operatorname{var}(f';[0,s]))ds,$$
$$t\in R^+.$$

By (1.11), (2.5), (2.20) and the fact that (1.12) implies that (1.22) holds there exist continuous functions h_1 and h_2 so that

$$\int_0^t\|u'_{\lambda,n}(s)\|ds\le h_1(t)\int_0^t b(t-s)\int_0^s\|u'_{\lambda,n}(\tau)\|d\tau ds+h_2(t),\qquad t\in R^+.$$

Since $b\in L^1_{loc}(R^+;R)$ we conclude from this inequality that

(2.21) $$\sup_{\lambda>0, n\geqq 1}\int_0^t \|u'_{\lambda,n}(s)\|ds<\infty, \qquad t\in R^+.$$

From (1.11), (2.4), (2.5), (2.14), (2.17)–(2.19) and (2.21) we deduce that

(2.22) $$\sup_{0\leqq s\leqq t, \lambda>0, n\geqq 1}\|B_\lambda(u_{\lambda,n}(s))\|<\infty, \qquad t\in R^+.$$

Now we can proceed to prove that the functions $u_{\lambda,n}$ converge when $\lambda\to 0$, $n\to\infty$. It is easily seen (since B_λ is Lipschitz-continuous) that $u_{\lambda,n}\to u_\lambda$ when $n\to\infty$, where u_λ is the solution of equation (1.19), but we want to show that the convergence is uniform with respect to λ. Let $m, n\geqq 1$ be integers and let $\lambda>0$ be arbitrary. By (2.14) we have

$$\begin{aligned}B_\lambda(u_{\lambda,n}(t))-B_\lambda(u_{\lambda,m}(t)) &= -b_n(0)^{-1}((u'_{\lambda,n}(t)-u'_{\lambda,m}(t))+p_n(0)(u_{\lambda,n}(t)-u_{\lambda,m}(t))\\ &+\int_0^t (u_{\lambda,n}(t-s)-u_{\lambda,m}(t-s))dp_n(s))+(-\omega+q_n(0))(u_{\lambda,n}(t)-u_{\lambda,m}(t))\\ &+\int_0^t (u_{\lambda,n}(t-s)-u_{\lambda,m}(t-s))dq_n(s)+(b_m(0)^{-1}-b_n(0)^{-1})u'_{\lambda,m}(t)\\ &+\int_0^t (b_m(0)^{-1}P_m(t-s)-b_n(0)^{-1}P_n(t-s))(u'_{\lambda,m}(s)-f'(s))ds\\ &\qquad \text{a.e. } t\in R^+.\end{aligned}$$

Hence it follows from (2.15) and (2.16) that

$$\begin{aligned}b_n(0)^{-1}d/dt\|u_{\lambda,n}(t)-u_{\lambda,m}(t)\| &+ b_n(0)^{-1}\int_0^t p_n(t-s)d/ds\,\|u_{\lambda,n}(s)-u_{\lambda,m}(s)\|ds\\ &\leqq (|\omega|+|q_n(0)|)\|u_{\lambda,n}(t)-u_{\lambda,m}(t)\|+\int_0^t \|u_{\lambda,n}(t-s)-u_{\lambda,m}(t-s)\|\,|dq_n(s)|\\ &+|b_m(0)^{-1}-b_n(0)^{-1}|\,\|u'_{\lambda,m}(t)\|\\ &+\int_0^t |b_m(0)^{-1}P_m(t-s)-b_n(0)^{-1}P_n(t-s)|\,\|u'_{\lambda,m}(s)-f'(s)\|ds\\ &\stackrel{\text{def}}{=} F_{\lambda,m,n}(t), \qquad \text{a.e. } t\in R^+.\end{aligned} \tag{2.23}$$

Proceeding in the same way as when we derived (2.20) we get

$$\|u_{\lambda,n}(t)-u_{\lambda,m}(t)\|\leqq\int_0^t b_n(t-s)F_{\lambda,m,n}(s)ds, \qquad t\in R^+.$$

From this inequality combined with (2.4), (2.5), (2.21) and (2.23) we are able to conclude that

$$\lim_{m,n\to\infty}\int_0^t \|u_{\lambda,n}(s)-u_{\lambda,m}(s)\|ds = 0 \quad \text{uniformly with respect to } \lambda \text{ for every } t\in R^+.$$

But since it follows from (2.13) and (2.22) that the functions $u_{\lambda,n}$ are equicontinuous on every interval $[0,t]$ we deduce that

$$\begin{aligned} &u_{\lambda,n}\to u_\lambda \text{ as } n\to\infty \text{ uniformly on } [0,T] \text{ for every } T>0 \text{ and} \\ &\qquad\qquad \text{uniformly with respect to } \lambda. \end{aligned} \tag{2.24}$$

We may apply [6, theorem 3] to equation (2.14) and it follows that for every n and every $T>0$

$$\lim_{\lambda,\mu\to 0}\|u_{\lambda,n}(t)-u_{\mu,n}(t)\| = 0 \quad \text{uniformly on } [0,T].$$

If we combine this result with (2.24) we see that there exists a function $u\in C(R^+;X)$ such that

$$u_\lambda\to u \quad \text{as} \quad \lambda\to 0 \quad \text{uniformly on } [0,T] \text{ for every } T>0. \tag{2.25}$$

From (2.21), (2.24) and (2.25) we deduce that $u\in BV_{\text{loc}}(R^+;X)$. To see that $u(t)\in\overline{D(B)}$, $t\in R^+$, we have only to note that $u_\lambda(t)=J_\lambda(u_\lambda(t))+\lambda B_\lambda(u_\lambda(t))$ where $J_\lambda(u_\lambda(t))=(I+\lambda B)^{-1}u_\lambda(t)\in D(B)$ and apply (2.22), (2.24) and (2.25).

In view of (2.22), (2.24) and the fact that X, and hence also $L^2(0,T;X)$, $T>0$, is reflexive there exists a function $w\in L^\infty_{\text{loc}}(R^+;X)$ such that

$$A_{\lambda_m}(u_{\lambda_m})\rightharpoonup w \ \text{(weakly) in } L^2(0,T;X) \text{ as } \lambda_m\to 0 \text{ for every } T>0 \tag{2.26}$$

for some sequence $\{\lambda_m\}$. By (1.19), (2.25) and (2.26) it is clear that (1.14) holds. If $\lim_{t\to 0+}a(t)<+\infty$ then it follows from (1.4)–(1.8), (1.11), (1.14) and (1.15) that u is locally Lipschitz-continuous on R^+ and hence differentiable a.e. To see that (1.16) holds we can proceed in the same manner as in the proof of [2, theorem III 2.2] (see also the proof of [7, theorem 2]). We have only to use (2.14), the equation one gets from (2.2) by letting $n\to\infty$, and to observe that

$$\int_0^t P_n(t-s)(u'_{\lambda,n}(s)-f'(s))ds\to\int_0^t (u(t-s)-u(t))dP(s)$$

$$+P(t)(u(t)-u(0))-\int_0^t P(t-s)f'(s)ds, \qquad n\to\infty, \quad \lambda\to 0, \quad t>0,$$

where $P(t) = b(0)(p(t) - q(t))$, $t > 0$.

In the case when $\lim_{t\to 0+} a(t) = +\infty$ we assume that X^* is locally uniformly convex. This implies that the duality mapping $F: X \to X^*$ defined by

$$F(x) = \{x^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|_*^2\}$$

is singlevalued and continuous. Recall that

$$[J_\lambda(u_\lambda(t)), B_\lambda(u_\lambda(t))] \in B, \qquad t \in R^+ \tag{2.27}$$

and that $J_\lambda(u_\lambda(t) = u_\lambda(t) + \lambda B_\lambda(u_\lambda(t))$. Therefore it follows from (2.22) and (2.25) that $J_\lambda(u_\lambda) \to u$ as $\lambda \to 0$ uniformly on $[0, T]$ for every $T > 0$. But then we also have for any function $v \in L^2_{\mathrm{loc}}(R^+; X)$ by the continuity of F

$$F(J_\lambda(u_\lambda) - v) \to F(u - v) \quad \text{as } \lambda \to 0 \text{ in } L^2(0, T; X^*) \text{ for every } T > 0. \tag{2.28}$$

As it is easily seen that the operator $\hat{B} \subset L^2(0, T; X) \times L^2(0, T; X)$ defined by $[v, z] \in \hat{B}$ if $[v(t), z(t)] \in B$ a.e. $t \in [0, T]$ is m-accretive and so also maximal accretive we conclude from (2.25)–(2.28) that (1.16) holds.

Finally we consider the question of uniqueness. Assume that we have two solutions. Subtracting one from the other we see that we have to consider the equation

$$v(t) + \int_0^t a(t-s) z(s) ds = 0, \qquad t \in R^+, \tag{2.29}$$

where $z \in L^\infty_{\mathrm{loc}}(R^+; X)$ and

$$[z(t), v(t)]_+ \geqq -|\omega| \, \|v(t)\|, \qquad \text{a.e. } t \in R^+. \tag{2.30}$$

Since we assume that $v \not\equiv 0$ we may without loss of generality assume that $v \not\equiv 0$ on $[0, T]$ for any $T > 0$.

Assume that $\lim_{t\to 0+} a(t) < +\infty$. Then we can differentiate (2.29) and solve z from the resulting equation and we obtain for any integer n

$$v'(t) + p_n(0) v(t) + \int_0^t v(t-s) dp_n(s) + b(0) z(t)$$

$$= \int_0^t (p_n(t-s) - b(0) p(t-s)) v'(s) ds + b(0) q(0) v(t)$$

$$+ b(0) \int_0^t v(t-s) dq(s), \qquad \text{a.e. } t \in R^+.$$

Combining this equation with (2.3), (2.4), (2.16), (2.29) and (2.30) and letting

$n \to \infty$ we conclude that we must have $v(t) = 0$, $t \in R^+$.

Next we assume that $\lim_{t \to 0+} a(t) = +\infty$. From (2.4), (2.29) and an integration by parts we see that

$$\int_0^t p(t-s)v(s)ds + \int_0^t z(s)ds + \int_0^t \int_0^{t-s} p(t-s-\tau)c'(\tau)d\tau \int_0^s z(\tau)d\tau ds = 0, \quad t \in R^+. \tag{2.31}$$

Let the function g be the solution of the equation

$$g(t) + \int_0^t g(t-s)\int_0^s p(s-\tau)c'(\tau)d\tau ds = \int_0^t p(t-s)c'(s)ds, \quad t \in R^+. \tag{2.32}$$

It follows from (1.9) and (2.32) that $g \in BV_{loc}(R^+; R)$ and from (2.31) and (2.32) that

$$\int_0^t p(t-s)v(s)ds + \int_0^t z(s)ds = \int_0^t g(t-s)\int_0^s p(s-\tau)v(\tau)d\tau ds, \quad t \in R^+$$

and so

$$d/dt \int_0^t p(t-s)v(s)ds + z(t) = \int_0^t \int_0^{t-s} p(t-s-\tau)v(\tau)d\tau dg(s), \quad \text{a.e. } t \in R^+. \tag{2.33}$$

We observe that the operator $f \to d/dt \int_0^t p(t-s)f(s) \overset{\text{def}}{=} L(f)$ is the inverse of the operator $f \to \int_0^t b(t-s)f(s)ds$. To prove that $L - (|\omega| + 1)I$ is accretive on $L^2(0, T; X)$ we have only to show that

$$\int_0^T |r_\mu(s)|ds \leqq (1 + (|\omega| + 1)\mu)^{-1} \qquad \text{for all } \mu > 0$$

where $r_\mu(t) = -R_1'\nu_\mu(t)$, $t \in R^+$ (see (2.6)). To see that this is the case we note that since r_μ is positive we have

$$\int_0^T |r_\mu(s)|ds \leqq \int_0^\infty r_\mu(s)ds \leqq \left(1 + \mu\left(\int_0^\infty b(s)ds\right)^{-1}\right)^{-1}$$

and since r_μ remains unchanged on $[0, T]$ if we change the values of b when $t > T$ (but so that (1.5)–(1.7) still hold) we may assume that $\int_0^\infty b(s)ds \leqq (|\omega| + 1)^{-1}$, if T is small enough.

Since X^* is locally uniformly convex it follows that

$$[x_1 + x_2, y]_+ = [x_1, y]_+ + [x_2, y]_+, \qquad x_1, x_2, y \in L^2(0, T; X),$$

($[\ ,\ ]_+$ defined on $L^2(0, T; X) \times L^2(0, T; X)$ as above with the norm of X replaced by that of $L^2(0, T; X)$). Therefore we can combine (2.30), (2.33) with the fact that $L - (|\omega| + 1)I$ is accretive on $L^2(0, T; X)$ if T is small enough and conclude that we must have $\int_0^t \|v(s)\|^2 ds = 0$ for small t. This implies that the solution is unique and the proof of Theorem 1 is completed.

3. Proofs of Theorems 2 and 3

We are first going to establish Theorem 3 for the equation (1.19), then we are able to prove Theorem 2 and finally we complete the proof of Theorem 3.

Assume that (1.23) and (1.24) hold and let $u_{1,\lambda,n}$, $u_{2,\lambda,n}$ and $u_{1,\lambda}$, $u_{2,\lambda}$ be the solutions of equations (2.13) and (1.19) respectively, corresponding to f_1 and f_2, and let $v_{\lambda,n} = u_{1,\lambda,n} - u_{2,\lambda,n}$ and $f = f_1 - f_2$. From (2.3) and (2.14)–(2.16) we obtain in the same manner as in the proof of Theorem 1

$$d/dt\|v_{\lambda,n}(t)\| + \int_0^t p_n(t-s)d/ds\,\|v_{\lambda,n}(s)\|ds$$

$$\leqq b_n(0)\Big((|\omega| + |q_n(0)|)\|v_{\lambda,n}(t)\| + \int_0^t \|v_{\lambda,n}(t-s)\|\,|dq_n(s)| + |q_n(t)|\,\|f(0)\|$$

$$+ \int_0^t |q_n(t-s)|\,\|f'(s)\|ds\Big) + \|f'(t)\| + \int_0^t p_n(t-s)\|f'(s)\|ds, \quad \text{a.e. } t \in R^+.$$

If we use (2.1) and integrate, then we get

$$\|v_{\lambda,n}(t)\| \leqq \|f(0)\| + \int_0^t \|f'(s)\|ds + \int_0^t b_n(t-s)\Big((|\omega| + |q_n(0)|)\|v_{\lambda,n}(s)\|$$

$$+ \int_0^s \|v_{\lambda,n}(s-\tau)\|\,|dq_n(\tau)| + |q_n(s)|\,\|f(0)\| + \int_0^s |q_n(s-\tau)|\,\|f'(\tau)\|d\tau\Big)ds,$$

$$t \in R^+.$$

It is straightforward to conclude from this integral inequality, (2.5) and (2.24) that there exists a nondecreasing, continuous function k on R^+, $k(0) = 1$ such that

$$(3.1)\quad \|u_{1,\lambda}(t)-u_{2,\lambda}(t)\| \leqq k(t)\left(\|f_1(0)-f_2(0)\|+\int_0^t \|f_1'(s)-f_2'(s)\|ds\right), \quad t\in R^+.$$

Now we proceed to the proof of Theorem 2. Let f satisfy (1.10) and (1.18). Then we can choose a sequence of functions $\{f_n\}$ that satisfy (1.10)–(1.12) such that

$$(3.2)\qquad f_n(0)\to f(0) \quad \text{as } n\to\infty,$$

$$(3.3)\qquad \int_0^T \|f_n'(s)-f'(s)\|ds \to 0 \quad \text{as } n\to\infty \text{ for every } T>0.$$

Let $u_{n,\lambda}$ be the solution of equation (1.19) corresponding to the function f_n. By (3.1)–(3.3) we know that the functions $u_{n,\lambda}$ converge uniformly on $[0,T]$ to u_λ as $n\to\infty$ for every $T>0$ and this convergence is uniform with respect to λ. On the other hand we know from the proof of Theorem 1 that for any fixed n, $u_{n,\lambda}$ converges uniformly on $[0,T]$ for every $T>0$ to a function in $C(R^+;\overline{D(B)})$ as $\lambda\to 0$. Combining these two observations we obtain (1.20) and (1.21). The remaining assertions of Theorem 2 have already been established in the proof of Theorem 1.

To complete the proof of Theorem 3 we have only to note that (1.25) follows from (1.21) and (3.1).

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